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WEAK ERROR IN NEGATIVE SOBOLEV SPACES FOR THE STOCHASTIC HEAT EQUATION

OMAR ABOURA

ABSTRACT. In this paper, we make another step in the study of weak error of the stochastic heat equation by considering norms as functional.

1. INTRODUCTION

Let (Ω, \mathcal{F}, P) a probability space and $T > 0$ a fixed time. $(W(t))_{t \geq 0}$ will be a cylindrical Brownian motion on $L^2(0, 1)$. We consider the stochastic heat equation, written in abstract form in $L^2(0, 1)$: $X(0) = 0$, for all $t \in [0, T]$ $X(t, 0) = X(t, 1) = 0$ and

$$dX(t) = \frac{1}{2} \frac{d^2}{dx^2} X(t) dt + dW(t). \quad (1.1)$$

It is well known that this equation admits a unique weak solution (from the analytical point of view).

Let $N \in \mathbb{N}^*$ and $h := T/N$. Consider $(t_k)_{0 \leq k \leq N}$ the uniform subdivision of $[0, T]$ defined by $t_k := kh$. We consider the implicit Euler scheme defined as follow:

$$X^N(t_{k+1}) = X^N(t_k) + h \frac{1}{2} \frac{d^2}{dx^2} X^N(t_{k+1}) + \Delta W(k+1), \quad (1.2)$$

where $\Delta W(k+1) = W(t_{k+1}) - W(t_k)$.

Let $f : L^2(0, 1) \rightarrow \mathbb{R}$ be a functional. The strong error is the study of $E |X^N(T) - X(T)|_{L^2(0,1)}^2$. The weak error is the study of $|Ef(X^N(T)) - Ef(X(T))|$ with respect to the time mesh h .

In [6], A. Debussche considers a more general stochastic equation and a more general functional than the one considered here. He obtains a weak error of order $1/2$, which is the double of that proved by [15] for the strong speed of convergence. The novelty of this paper is to prove that for the square of the norm the weak error is better than $1/2$ in negative Sobolev spaces.

2. PRELIMINARIES AND MAIN RESULT

Notations. We collect here some of the notations used through the paper. $\langle \cdot, \cdot \rangle_{L^2(0,1)}$ is the inner product in $L^2(0, 1)$, $H_0^1(0, 1)$ is the Sobolev space of functions f in $L^2(0, 1)$ vanishing in 0 and 1 with first derivatives in $L^2(0, 1)$, $H^2(0, 1)$ is the Sobolev space of functions f in $L^2(0, 1)$ with first and second derivatives in $L^2(0, 1)$. Finally, for $m = 1, 2, \dots$, let $(e_m(x) = \sqrt{2} \sin(m\pi x))$ and $\lambda_m = \frac{1}{2}(\pi m)^2$ denote the eigenfunction and eigenvalues of $-\Delta$ with Dirichlet boundary conditions on $(0, 1)$.

An $L^2(0, 1)$ -valued stochastic process $(X(t))_{t \in [0, T]}$ is said to be a solution of (1.1) if: $X(0) = 0$ and for all $g \in H_0^1(0, 1) \cap H^2(0, 1)$ we have

$$\langle X(t), g \rangle_{L^2(0,1)} = \int_0^t \langle X(s), \frac{1}{2} \frac{d^2}{dx^2} g \rangle_{L^2(0,1)} ds + \langle W(t), g \rangle_{L^2(0,1)}.$$

It is well known that (1.1) admits a unique solution: see [4]. Then $(e_m)_{m \geq 1}$ is a complete orthonormal basis of $L^2(0, 1)$. If we denote by $\lambda_m := \frac{1}{2}(\pi m)^2$, $W_{\lambda_m}(t) := \langle W(t), e_m \rangle_H$ and $X_{\lambda_m}(t)$ denote the solution of the evolution equation: $X_{\lambda_m}(0) = 0$ and for $t > 0$:

$$dX_{\lambda_m}(t) = -\lambda_m X_{\lambda_m}(t)dt + dW_{\lambda_m}(t).$$

Then the processes $(X_{\lambda_m}(\cdot))_{m \geq 1}$ are independent and $X(t) = \sum_{m \geq 1} X_{\lambda_m}(t)e_m$ for all $t \geq 0$.

A sequence of $L^2(0, 1)$ -valued $(X^N(t_k))_{k=0, \dots, N}$ is said to be a solution of (1.2) if: $X^N(t_0) = 0$ and for all $k = 0, \dots, N-1$ and for all $g \in H_0^1(0, 1) \cap H^2(0, 1)$ we have

$$\begin{aligned} \langle X^N(t_{k+1}), g \rangle_{L^2(0,1)} &= \langle X^N(t_k), g \rangle_{L^2(0,1)} + h \langle X^N(t_{k+1}), \frac{1}{2} \frac{d^2}{dx^2} g \rangle_{L^2(0,1)} \\ &\quad + \langle \Delta W(k+1), g \rangle_{L^2(0,1)}. \end{aligned}$$

It is well known that (1.2) has a unique solution and there exists a constant $C > 0$, independent of N , such that $E |X^N(T) - X(T)|_{L^2(0,1)}^2 \leq Ch^{\frac{1}{2}}$ where $h = T/N$. Now if we denote by $(X_{\lambda_m}^N(t_k))_{k=0, \dots, N}$ the solution of: $X_{\lambda_m}^N(t_0) = 0$ and for $k = 0, \dots, N-1$

$$X_{\lambda_m}^N(t_{k+1}) = X_{\lambda_m}^N(t_k) - \lambda_m h X_{\lambda_m}^N(t_{k+1}) + W_{\lambda_m}(k+1).$$

The random vectors $(X_{\lambda_m}^N(t_k), k = 0, \dots, N)_{m=1,2,\dots}$ are independent and $X^N(t_k) = \sum_{m \geq 1} X_{\lambda_m}^N(t_k)e_m$.

Let $p \geq 0$; we define the spaces H^{-p} as the completion of $L^2(0, 1)$ for the topology induced by the norm $|u|_{H^{-p}}^2 := \sum_{m \geq 1} \lambda_m^{-p} \langle u, e_m \rangle_H^2$. The following theorem improves the speed of convergence of X^N to X for negative Sobolev spaces.

Theorem 2.1. *Suppose that $h < 1$ and let $p \in [0, \frac{1}{2})$. There exists a constant $C > 0$, independent of N , such that*

$$\left| E |X^N(T)|_{H^{-p}}^2 - E |X(T)|_{H^{-p}}^2 \right| \leq Ch^{p+\frac{1}{2}}.$$

3. PROOF OF THE THEOREM 2.1

The proof of the theorem will be done in several steps. First we recall the weak error of the Ornstein-Uhlenbeck process. Secondly we prove some technical lemmas. Then we decompose the weak error and analyse each term of these decomposition.

3.1. Weak error of the Ornstein-Uhlenbeck process. Let $\lambda > 0$, $(W_\lambda(t))_{t \geq 0}$ be a one dimensional Brownian motion and $(X_\lambda(t))_{t \geq 0}$ be the Ornstein-Uhlenbeck process solution of the following stochastic differential equation: $X_\lambda(0) = x \in \mathbb{R}$ and

$$dX_\lambda(t) = -\lambda X_\lambda(t)dt + dW_\lambda(t). \quad (3.1)$$

In this step, we study two properties associated with this process: the Kolmogorov equation and the implicit Euler scheme.

Let $(X_\lambda^{t,x}(s))_{t \leq s \leq T}$ be the solution of (3.1) starting from x at time t . It is well known that $X_\lambda^{t,x}(T)$ is a normal random variable:

$$X_\lambda^{t,x}(T) \sim \mathcal{N} \left(e^{-\lambda(T-t)}x, \frac{1 - e^{-2\lambda(T-t)}}{2\lambda} \right).$$

For $t \in [0, T]$ and $x \in \mathbb{R}$ set $u_\lambda(t, x) := E \left| X_\lambda^{t,x}(T) \right|^2$. Then u_λ is the solution of the following partial differential equation, called Kolmogorov equation: for all $x \in \mathbb{R}$, $u_\lambda(T, x) = |x|^2$ and for all $(t, x) \in [0, T) \times \mathbb{R}$

$$-\frac{\partial}{\partial t}u(t, x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t, x) - \lambda x \frac{\partial}{\partial x}u(t, x). \quad (3.2)$$

Since $X_\lambda^{t,x}(T)$ has a normal law, we can write u_λ explicitly:

$$u_\lambda(t, x) = \frac{1 - e^{-2\lambda(T-t)}}{2\lambda} + e^{-2\lambda(T-t)}x^2. \quad (3.3)$$

With this expression we see that $u_\lambda \in C^{1,2}([0, T] \times \mathbb{R})$ and we have the following derivatives:

$$\frac{\partial}{\partial x}u_\lambda(t, x) = 2e^{-2\lambda(T-t)}x, \quad (3.4)$$

$$\frac{\partial^2}{\partial x^2}u_\lambda(t, x) = 2e^{-2\lambda(T-t)}, \quad (3.5)$$

$$\frac{\partial}{\partial t}u_\lambda(t, x) = -e^{-2\lambda(T-t)} + 2\lambda e^{-2\lambda(T-t)}x^2, \quad (3.6)$$

$$\frac{\partial^2}{\partial t \partial x}u_\lambda(t, x) = 4\lambda e^{-2\lambda(T-t)}x. \quad (3.7)$$

The implicit Euler scheme for the Ornstein-Uhlenbeck equation (3.1) starting from 0 at time t_0 , is defined as follow: $X_\lambda^N(t_0) = 0$ and for $k = 0, \dots, N-1$

$$X_\lambda^N(t_{k+1}) = X_\lambda^N(t_k) - \lambda h X_\lambda^N(t_{k+1}) + \Delta W_\lambda(k+1), \quad (3.8)$$

where $\Delta W_\lambda(k+1) = W_\lambda(t_{k+1}) - W_\lambda(t_k)$. Since we have the following equation

$$X_\lambda^N(t_{k+1}) = \frac{1}{1 + \lambda h} X_\lambda^N(t_k) + \frac{1}{1 + \lambda h} \Delta W_\lambda(k+1), \quad (3.9)$$

we see that the scheme is well defined.

Lemma 3.1. For $k = 1, \dots, N$ we have $X_\lambda^N(t_k) = \sum_{j=0}^{k-1} \frac{\Delta W_\lambda(k-j)}{(1 + \lambda h)^{j+1}}$.

Proof. We proceed by induction. If $k = 1$, we have $X_\lambda^N(t_1) = \frac{1}{1 + \lambda h} \Delta W_\lambda(1)$. Suppose the result true until k . Using (3.9), we have

$$\begin{aligned} X_\lambda^N(t_{k+1}) &= \sum_{j=0}^{k-1} \frac{\Delta W_\lambda(k-j)}{(1 + \lambda h)^{j+2}} + \frac{1}{1 + \lambda h} \Delta W_\lambda(k+1) \\ &= \sum_{l=1}^k \frac{\Delta W_\lambda(k+1-l)}{(1 + \lambda h)^{l+1}} + \frac{1}{(1 + \lambda h)^{0+1}} \Delta W_\lambda(k+1-0), \end{aligned}$$

which concludes the proof. \square

Lemma 3.2. For all $k = 0, \dots, N$, we have the following bound $E |X_\lambda^N(t_k)|^2 \leq \frac{1}{2\lambda}$.

Proof. Using the independence of the increments of the Brownian motion and Lemma 3.1, we have

$$E |X_\lambda^N(t_k)|^2 = \sum_{j=0}^{k-1} \frac{1}{(1 + \lambda h)^{2(j+1)}} E |\Delta W_\lambda(k-j)|^2 = h \sum_{j=0}^{k-1} \frac{1}{(1 + \lambda h)^{2(j+1)}}.$$

Let $a := 1/(1 + \lambda h)^2$; we deduce that $E |X_\lambda^N(t_k)|^2 = ha \frac{1-a^k}{1-a}$. Simple computations yield $ha/(1-a) = 1/(2\lambda + \lambda^2 h)$, which implies

$$E |X_\lambda^N(t_k)|^2 = \frac{1}{2\lambda + \lambda^2 h} \left(1 - \frac{1}{(1 + \lambda h)^{2k}} \right).$$

This concludes the proof. \square

For $t \geq 0$, we denote $\mathcal{F}_t^\lambda := \sigma(W_\lambda(s), s \leq t)$ and $D_\lambda^{1,2}$ the Malliavin Sobolev space with respect to W_λ .

Lemma 3.3. *For all $k = 1, \dots, N$, we have $X_\lambda^N(t_k) \in D_\lambda^{1,2} \cap L^2(\mathcal{F}_{t_k}^\lambda)$.*

Proof. This is a consequence of Lemma 3.1, the fact that $L^2(\mathcal{F}_{t_k}^\lambda)$ and $D_\lambda^{1,2}$ are linear space and for all $j = 0, \dots, k-1$, $\Delta W_\lambda(k-j) \in D_\lambda^{1,2} \cap L^2(\mathcal{F}_{t_k}^\lambda)$. \square

As usual in the study of weak error, we need to use a continuous process that interpolates the Euler scheme. The interpolation process that we use was introduced in [1]. We recall its construction and prove some of its properties.

Let $k \in \{0, \dots, N-1\}$ be fixed. In order to interpolate the scheme between the points $(t_k, X_\lambda^N(t_k))$ and $(t_{k+1}, X_\lambda^N(t_{k+1}))$, we define the process as follows: for $t \in [t_k, t_{k+1}]$, set

$$X_\lambda^N(t) := X_\lambda^N(t_k) - \lambda E(X_\lambda^N(t_{k+1}) | \mathcal{F}_t) (t - t_k) + W_\lambda(t) - W_\lambda(t_k). \quad (3.10)$$

In the sequel, we will use the following processes: for $t \in [t_k, t_{k+1}]$

$$\beta_\lambda^{k,N}(t) := -\lambda E(X_\lambda^N(t_{k+1}) | \mathcal{F}_t), \quad (3.11)$$

$$z_\lambda^{k,N}(t) := -\lambda E(D_t X_\lambda^N(t_{k+1}) | \mathcal{F}_t), \quad (3.12)$$

$$\gamma_\lambda^{k,N}(t) := 1 + (t - t_k) z_\lambda^{k,N}(t). \quad (3.13)$$

The next lemma relates the above processes.

Lemma 3.4. *Let $k = 0, \dots, N-1$. For $t \in [0, T]$, we have*

$$\begin{aligned} d\beta_\lambda^{k,N}(t) &= z_\lambda^{k,N}(t) dW_\lambda(t), \quad z_\lambda^{k,N}(t) = -\frac{\lambda}{1 + \lambda h}, \\ \gamma_\lambda^{k,N}(t) &= 1 - (t - t_k) \frac{\lambda}{1 + \lambda h}, \quad dX_\lambda^N(t) = \beta_\lambda^{k,N}(t) dt + \gamma_\lambda^{k,N}(t) dW_\lambda(t). \end{aligned}$$

Proof. Using the Clark-Ocone formula and Lemma 3.3, we have

$$X_\lambda^N(t_{k+1}) = E(X_\lambda^N(t_{k+1}) | \mathcal{F}_t) + \int_t^{t_{k+1}} E(D_s X_\lambda^N(t_{k+1}) | \mathcal{F}_s) dW_\lambda(s).$$

Multiplying by $(-\lambda)$, we deduce

$$-\lambda X_\lambda^N(t_{k+1}) = \beta_\lambda^{k,N}(t) + \int_t^{t_{k+1}} z_\lambda^{k,N}(s) dW_\lambda(s),$$

which gives the first identity. Applying the Malliavin derivative to (3.9), we have for $s \in [t_k, t_{k+1}]$ $D_s X_\lambda^N(t_{k+1}) = \frac{1}{1 + \lambda h}$. Multiplying by $(-\lambda)$, we deduce the second and third equalities.

Finally, Itô's formula gives us

$$d\left((t - t_k) \beta_\lambda^{k,N}(t)\right) = (t - t_k) z_\lambda^{k,N}(t) dW_\lambda(t) + \beta_\lambda^{k,N}(t) dt,$$

which concludes the proof. \square

Lemma 3.5. *Let $k \in \{0, \dots, N-1\}$. For any $s \in [t_k, t_{k+1}]$, we have*

$$E \left| \beta_\lambda^{k,N}(s) \right|^2 \leq 2\lambda, \quad E \left| X_\lambda^N(s) \right|^2 \leq \frac{1}{2\lambda} + h, \quad E \beta_\lambda^{k,N}(s) X_\lambda^N(s) \leq 1.$$

Proof. Applying the conditionnal expectation with respect to \mathcal{F}_s on both sides of (3.9) for $s \in [t_k, t_{k+1})$ we have

$$E \left(X_\lambda^N(t_{k+1}) | \mathcal{F}_s \right) = \frac{1}{1 + \lambda h} \left[X_\lambda^N(t_k) + (W_\lambda(s) - W_\lambda(t_k)) \right].$$

Multiplying by $(-\lambda)$ and using (3.11), we obtain

$$\beta_\lambda^{k,N}(s) = -\frac{\lambda}{1 + \lambda h} X_\lambda^N(t_k) - \frac{\lambda}{1 + \lambda h} (W_\lambda(s) - W_\lambda(t_k)). \quad (3.14)$$

The independence of \mathcal{F}_{t_k} and $W_\lambda(s) - W_\lambda(t_k)$ yields

$$E \left| \beta_\lambda^{k,N}(s) \right|^2 = \frac{\lambda^2}{(1 + \lambda h)^2} E \left| X_\lambda^N(t_k) \right|^2 + \frac{\lambda^2}{(1 + \lambda h)^2} (s - t_k).$$

Using Lemma 3.2, we deduce

$$E \left| \beta_\lambda^{k,N}(s) \right|^2 \leq \frac{\lambda}{2(1 + \lambda h)^2} + \frac{\lambda^2 h}{(1 + \lambda h)^2},$$

which proves the first upper estimate.

Using (3.10) and (3.14), we have for $s \in [t_k, t_{k+1}]$

$$X_\lambda^N(s) = \left(1 - \frac{\lambda(s - t_k)}{1 + \lambda h} \right) \left[X_\lambda^N(t_k) + (W_\lambda(s) - W_\lambda(t_k)) \right]. \quad (3.15)$$

Taking the expectation of the square and using the independence of \mathcal{F}_{t_k} and $W_\lambda(s) - W_\lambda(t_k)$, we have

$$E \left| X_\lambda^N(s) \right|^2 = \left(1 - \frac{\lambda(s - t_k)}{1 + \lambda h} \right)^2 \left[E \left| X_\lambda^N(t_k) \right|^2 + (s - t_k) \right] \leq E \left| X_\lambda^N(t_k) \right|^2 + h \leq \frac{1}{2\lambda} + h,$$

where the last upper estimates follows from Lemma 3.2.

Multiplying (3.14) and (3.15), taking expectation we obtain

$$E \left(X_\lambda^N(s) \beta_\lambda^{k,N}(s) \right) = \frac{-\lambda}{1 + \lambda h} \left(1 - \frac{\lambda(s - t_k)}{1 + \lambda h} \right) \left[E \left| X_\lambda^N(t_k) \right|^2 + (s - t_k) \right].$$

Using Lemma 3.2, we deduce

$$\left| E \left(X_\lambda^N(s) \beta_\lambda^{k,N}(s) \right) \right| \leq \frac{\lambda}{1 + \lambda h} \frac{1}{2\lambda} + \frac{\lambda h}{1 + \lambda h}.$$

This concludes the proof. \square

3.2. Some useful analytical lemmas. We at first give a precise upper bound of a series defined in terms of the eigenvalues of the Laplace operator with Dirichlet boundary conditions.

Lemma 3.6. *Let $p \in [0, \frac{1}{2})$. There exists a constant $C > 0$, such that for all $\alpha > 0$, we have*

$$\sum_{m \geq 1} \lambda_m^{-p} e^{-2\lambda_m \alpha} \leq C \alpha^{p-\frac{1}{2}}$$

Proof. The function $(x \in \mathbb{R}_+ \mapsto x^{-2p}e^{-2x^2\alpha})$ is decreasing. So by comparison, we obtain

$$\sum_{m \geq 1} m^{-2p} e^{-2m^2\alpha} \leq \int_0^\infty x^{-2p} e^{-2x^2\alpha} dx \leq \alpha^{p-\frac{1}{2}} \int_0^\infty y^{-2p} e^{-2y^2} dy = C\alpha^{p-\frac{1}{2}}.$$

Since $\lambda_m = \frac{1}{2}(\pi m)^2$, we deduce the desired upper estimate. \square

Lemma 3.7. *Let $q > 0$. There exists a constant $C > 0$, such that for all $\alpha > 0$*

$$\sum_{m \geq 1} \lambda_m^q e^{-\lambda_m \alpha} \leq C \left(1 + \frac{1}{\alpha^{q+\frac{1}{2}}} \right).$$

Proof. Let $f(x) = x^{2q}e^{-x^2\alpha}$. His derivatives is given by $f'(x) = 2x^{2q-1}e^{-x^2\alpha}(q - \alpha x^2)$.
Case 1: $\alpha > q/4$. Then f is decreasing on $[2, \infty)$ and a standard comparison argument yields

$$\begin{aligned} \sum_{m \geq 1} m^{2q} e^{-m^2\alpha} &\leq e^{-\alpha} + 4^q e^{-4\alpha} + \sum_{m \geq 3} \int_{m-1}^m x^{2q} e^{-x^2\alpha} dx \\ &\leq C + \int_0^\infty x^{2q} e^{-x^2\alpha} dx \\ &\leq C + \alpha^{-q-\frac{1}{2}} \int_0^\infty y^{2q} e^{-y^2} dy \\ &\leq C(1 + \alpha^{-q-\frac{1}{2}}). \end{aligned}$$

Case 2: $\alpha \leq q/4$. The function f is increasing on $[0, \sqrt{q/\alpha}]$. So for each $m = 1, \dots, [\sqrt{q/\alpha}] - 1$, we have

$$m^{2q} e^{-m^2\alpha} \leq \int_m^{m+1} x^{2q} e^{-x^2\alpha} dx.$$

On the interval $[\sqrt{q/\alpha}, \infty)$, f is decreasing. So for each integer $m \geq [\sqrt{q/\alpha}] + 2$, we have

$$m^{2q} e^{-m^2\alpha} \leq \int_{m-1}^m x^{2q} e^{-x^2\alpha} dx.$$

The above upper estimates yield

$$\begin{aligned} \sum_{m \geq 1} m^{2q} e^{-m^2\alpha} &\leq \sum_{m \leq [\sqrt{q/\alpha}] - 1} \int_m^{m+1} x^{2q} e^{-x^2\alpha} dx + \sum_{m \geq [\sqrt{q/\alpha}] + 2} \int_{m-1}^m x^{2q} e^{-x^2\alpha} dx \\ &\quad + \sum_{m \in \{[\sqrt{q/\alpha}], [\sqrt{q/\alpha}] + 1\}} m^{2q} e^{-m^2\alpha} \\ &\leq \int_0^\infty x^{2q} e^{-x^2\alpha} dx + \sum_{m \in \{[\sqrt{q/\alpha}], [\sqrt{q/\alpha}] + 1\}} m^{2q} e^{-m^2\alpha} \\ &\leq C\alpha^{-q-\frac{1}{2}} + \sum_{m \in \{[\sqrt{q/\alpha}], [\sqrt{q/\alpha}] + 1\}} m^{2q} e^{-m^2\alpha} \end{aligned}$$

Now we study each term of the sum in the right hand side. Since $q \geq \alpha$, we have

$$\left[\sqrt{\frac{q}{\alpha}} \right]^{2q} e^{-[\sqrt{q/\alpha}]^2 \alpha} \leq \left(\frac{q}{\alpha} \right)^q \leq \left(\frac{q}{\alpha} \right)^{q+\frac{1}{2}} \leq C\alpha^{-q-\frac{1}{2}}.$$

For the second term, we remark that since $q \geq \alpha \left[\sqrt{\frac{q}{\alpha}} \right] + 1 \leq 2 \left[\sqrt{\frac{q}{\alpha}} \right] \leq 2\sqrt{\frac{q}{\alpha}}$. This implies

$$\left(\left[\sqrt{\frac{q}{\alpha}} \right] + 1 \right)^{2q} e^{-([\sqrt{\frac{q}{\alpha}}]+1)^2 \alpha} \leq \left(2\sqrt{\frac{q}{\alpha}} \right)^{2q} \leq C \alpha^{-q-\frac{1}{2}}.$$

Therefore, in both cases we obtain

$$\sum_{m \geq 1} m^{2q} e^{-m^2 \alpha} \leq C \left(1 + \frac{1}{\alpha^{q+\frac{1}{2}}} \right).$$

Since $\lambda_m = \frac{1}{2}(\pi m)^2$, the proof is complete. \square

Lemma 3.8. *Let $p \in [0, \frac{1}{2})$ and $n \in \mathbb{N}^*$. Let $(v(k, m))_{(k, m) \in \{0, \dots, N-2\} \times \mathbb{N}^*}$ be a sequence such that for all $k \in \{0, \dots, N-2\}$ and $m \geq 1$, we have*

$$0 \leq v(k, m) \leq \lambda_m^{n-p} h^{n+1} e^{-2\lambda_m(T-t_{k+1})}.$$

Then, there exists a constant $C > 0$, independent of N , such that

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} v(k, m) \leq C h^{p+\frac{1}{2}}.$$

Proof. First we remark that $T - t_{k+1} = h(N - k - 1)$. Using Lemma 3.7, we deduce the existence of C depending on n and p , but independent of N , such that for $k = 0, \dots, N-2$:

$$\begin{aligned} \sum_{m \geq 1} v(k, m) &\leq C h^{n+1} \left(1 + \frac{1}{h^{n-p+\frac{1}{2}} (N - k - 1)^{n-p+\frac{1}{2}}} \right) \\ &\leq C \left(h^{n+1} + \frac{h^{p+\frac{1}{2}}}{(N - k - 1)^{n-p+\frac{1}{2}}} \right). \end{aligned}$$

Therefore, there exists a constant C as above such that

$$\begin{aligned} \sum_{m \geq 1} \sum_{k=0}^{N-2} v(k, m) &\leq C \left(h^n + h^{p+\frac{1}{2}} \sum_{k=0}^{N-2} \frac{1}{(N - k - 1)^{n-p+\frac{1}{2}}} \right) \\ &\leq C \left(h^n + h^{p+\frac{1}{2}} \sum_{l=1}^{N-1} \frac{1}{l^{n-p+\frac{1}{2}}} \right) \leq C h^{p+\frac{1}{2}}, \end{aligned}$$

which concludes the proof. \square

3.3. Decomposition of the weak error. We follow the classical decomposition introduced in [16]. The definition of $u_\lambda(t, x)$ in section 3.1 yields

$$\begin{aligned} E |X^N(T)|_{H^{-p}}^2 - E |X(T)|_{H^{-p}}^2 &= \sum_{m \geq 1} \lambda_m^{-p} \left(E |X_{\lambda_m}^N(T)|^2 - E |X_{\lambda_m}(T)|^2 \right) \\ &= \sum_{m \geq 1} \lambda_m^{-p} \left(E u_{\lambda_m}(T, X_{\lambda_m}^N(T)) - u_{\lambda_m}(0, X_{\lambda_m}^N(0)) \right). \end{aligned}$$

Let $\delta^N(k, m) := \lambda_m^{-p} (E u_{\lambda_m}(t_{k+1}, X_{\lambda_m}^N(t_{k+1})) - E u_{\lambda_m}(t_k, X_{\lambda_m}^N(t_k)))$; then

$$E |X^N(T)|_{H^{-p}}^2 - E |X(T)|_{H^{-p}}^2 = \sum_{m \geq 1} \sum_{k=0}^{N-1} \delta^N(k, m).$$

Note that using Lemmas 3.3, 3.4 and (3.4) we deduce that for any $k = 0, \dots, N-1$

$$E \int_{t_k}^{t_{k+1}} \left| \gamma_{\lambda}^{k,N}(t) \frac{\partial u}{\partial x}(t, X_{\lambda}^N(t)) \right|^2 dt < \infty.$$

From now, we do not justify that the stochastic integral are centered. Itô's formula and Lemma 3.4, we imply that for $k = 0, \dots, N-1$

$$\begin{aligned} \delta^N(k, m) &= \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} u_{\lambda_m} + \beta_{\lambda_m}^{k,N}(t) \frac{\partial}{\partial x} u_{\lambda_m} + \frac{1}{2} \left| \gamma_{\lambda_m}^{k,N}(t) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (t, X_{\lambda_m}^N(t)) dt \\ &= \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left\{ I_{\lambda_m}^{k,N}(t) + \frac{1}{2} J_{\lambda_m}^{k,N}(t) \right\} dt, \end{aligned}$$

where

$$I_{\lambda_m}^{k,N}(t) := \left(\beta_{\lambda_m}^{k,N}(t) + \lambda_m X_{\lambda_m}^N(t) \right) \frac{\partial}{\partial x} u_{\lambda_m}(t, X_{\lambda_m}^N(t)), \quad (3.16)$$

$$J_{\lambda_m}^{k,N}(t) := \left(\left| \gamma_{\lambda_m}^{k,N}(t) \right|^2 - 1 \right) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(t, X_{\lambda_m}^N(t)). \quad (3.17)$$

This yields the following decomposition:

$$\begin{aligned} E |X^N(T)|_{H^{-p}}^2 - E |X(T)|_{H^{-p}}^2 &= \sum_{m \geq 1} \delta^N(N-1, m) + \sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} I_{\lambda_m}^{k,N}(t) dt \\ &\quad + \frac{1}{2} \sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} J_{\lambda_m}^{k,N}(t) dt. \end{aligned} \quad (3.18)$$

Now we study each term of this decomposition.

Lemma 3.9. *There exists a constant C , independant of N , such that*

$$\sum_{m \geq 1} |\delta^N(N-1, m)| \leq Ch^{p+\frac{1}{2}}.$$

This study is similar to the third step of [6], page 97.

Proof. Using the definition of $u_{\lambda_m}(t, x)$ (3.3) and (3.9), we have

$$\begin{aligned} u_{\lambda_m}(t_N, X_{\lambda_m}^N(t_N)) &= |X_{\lambda_m}^N(t_N)|^2 = \frac{1}{(1 + \lambda_m h)^2} |X_{\lambda_m}^N(t_{N-1}) + \Delta W_m(N)|^2, \\ u_{\lambda_m}(t_{N-1}, X_{\lambda_m}^N(t_{N-1})) &= \frac{1 - e^{-2\lambda_m h}}{2\lambda_m} + e^{-2\lambda_m h} |X_{\lambda_m}^N(t_{N-1})|^2. \end{aligned}$$

By independence between $\Delta W_m(N)$ and $X_{\lambda_m}^N(t_{N-1})$, we have

$$\begin{aligned} \delta^N(N-1, m) &= \lambda_m^{-p} \left\{ \frac{1}{(1 + \lambda_m h)^2} - e^{-2\lambda_m h} \right\} E |X_{\lambda_m}^N(t_{N-1})|^2 \\ &\quad + \frac{h}{\lambda_m^p (1 + \lambda_m h)^2} - \frac{1 - e^{-2\lambda_m h}}{2\lambda_m^{1+p}}. \end{aligned}$$

Let $\delta_1(\lambda_m) := \frac{1 - 2e^{-2\lambda_m h}}{2\lambda_m^{1+p}}$, $\delta_2(\lambda_m) := \frac{h}{\lambda_m^p (1 + \lambda_m h)^2}$, and

$$\delta_3(\lambda_m) := \lambda_m^{-p} \left\{ \frac{1}{(1 + \lambda_m h)^2} - e^{-2\lambda_m h} \right\} E |X_{\lambda_m}^N(t_{N-1})|^2.$$

With these notations we have

$$\delta^N(N-1, m) \leq \delta_1(\lambda_m) + \delta_2(\lambda_m) + \delta_3(\lambda_m).$$

First, we study $\delta_1(\lambda_m)$. Since $\frac{1-e^{-2\lambda h}}{2\lambda} = \int_0^h e^{-2\lambda x} dx$, using Lemma 3.6, we obtain

$$\sum_{m \geq 1} \delta_1(\lambda_m) = \int_0^h \sum_{m \geq 1} \lambda_m^{-p} e^{-2\lambda_m x} dx \leq C \int_0^h x^{p-\frac{1}{2}} dx = Ch^{p+\frac{1}{2}}. \quad (3.19)$$

Now we study $\delta_2(\lambda_m)$. Since $(x \in [0, \infty) \mapsto x^{-2p}(1+x^2h)^2)$ is decreasing, we have for $p \in [0, \frac{1}{2})$

$$\sum_{m \geq 1} \delta_2(\lambda_m) \leq Ch \int_0^\infty \frac{1}{x^{2p}(1+x^2h)^2} dx \leq Ch^{p+\frac{1}{2}} \int_0^\infty \frac{y^{-2p}}{(1+y^2)^2} dy \leq Ch^{p+\frac{1}{2}}. \quad (3.20)$$

Finally, we study $\delta_3(\lambda_m)$. Using Lemma 3.2, we have

$$\delta_3(\lambda_m) \leq \lambda_m^{-p} \left\{ \frac{1}{(1+\lambda_m h)^2} - e^{-2\lambda_m h} \right\} \frac{1}{2\lambda_m}.$$

Since $\frac{1}{(1+\lambda h)^2} - e^{-2\lambda h} = 2\lambda \int_0^h \left\{ e^{-2\lambda x} - \frac{1}{(1+\lambda x)^3} \right\} dx$, we have

$$\delta_3(\lambda_m) \leq \lambda_m^{-p} \int_0^h \left\{ e^{-2\lambda_m x} + \frac{1}{(1+\lambda_m x)^3} \right\} dx.$$

Using Lemma 3.6, we have for $p \in [0, \frac{1}{2})$

$$\sum_{m \geq 1} \lambda_m^{-p} \int_0^h e^{-2\lambda_m x} dx \leq C \int_0^h x^{p-\frac{1}{2}} dx \leq Ch^{p+\frac{1}{2}}.$$

Now since for $x \geq 0$ the map $(y \in \mathbb{R}_+ \mapsto y^{-2p}(1+y^2x)^{-3})$ is decreasing, we have for $p \in [0, \frac{1}{2})$

$$\sum_{m \geq 1} \frac{\lambda_m^{-p}}{(1+\lambda_m x)^3} \leq C \int_0^\infty \frac{1}{y^{2p}(1+y^2x)} dy \leq Cx^{p-\frac{1}{2}} \int_0^\infty \frac{1}{z^{2p}(1+z^2)^3} dz \leq Cx^{p-\frac{1}{2}},$$

and hence Fubini's theorem yields

$$\sum_{m \geq 1} \int_0^h \frac{\lambda_m^{-p}}{(1+\lambda_m x)^3} dx \leq C \int_0^h x^{p-\frac{1}{2}} dx \leq Ch^{p+\frac{1}{2}}.$$

The above inequalities imply $\sum_{m \geq 1} \delta_3(\lambda_m) \leq Ch^{p+\frac{1}{2}}$. This inequality, (3.19) and (3.20) give the stated upper estimate. \square

Lemma 3.10. *There exists a constant $C > 0$, independent of N , such that*

$$\sum_{m \geq 1} \sum_{k=0}^{N-1} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} |J_{\lambda_m}^{k,N}(t)| dt \leq Ch^{p+\frac{1}{2}}.$$

Proof. Using Lemma 3.4, we have

$$\left| \gamma_{\lambda_m}^{k,N}(t) \right|^2 - 1 = -\frac{2(t-t_k)\lambda_m}{1+\lambda_m h} + \frac{|t-t_k|^2 \lambda_m^2}{(1+\lambda_m h)^2}.$$

Using (3.5) and (3.17), we have

$$\lambda_m^{-p} E \int_{t_k}^{t_{k+1}} |J_{\lambda_m}^{k,N}(t)| dt \leq C (\lambda_m^{1-p} h^2 + \lambda_m^{2-p} h^3) e^{-2\lambda_m(T-t_{k+1})}.$$

Lemma 3.8 concludes the proof. \square

Lemma 3.11. *There exists a constant $C > 0$, independant of N , such that*

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| I_{\lambda_m}^{k,N}(t) \right| dt \leq Ch^{p+\frac{1}{2}}.$$

Proof. Let $I_{1,\lambda_m}^{k,N}(t) := E\beta_{\lambda_m}^{k,N}(t) \frac{\partial}{\partial x} u_{\lambda_m}(t, X_{\lambda_m}^N(t)) + E\lambda_m X_{\lambda_m}^N(t_{k+1}) \frac{\partial}{\partial x} u_{\lambda_m}(t_{k+1}, X_{\lambda_m}^N(t_{k+1}))$ and $I_{2,\lambda_m}^{k,N}(t) := -\lambda_m E X_{\lambda_m}^N(t_{k+1}) \frac{\partial}{\partial x} u_{\lambda_m}(t_{k+1}, X_{\lambda_m}^N(t_{k+1})) + \lambda_m E X_{\lambda_m}^N(t) \frac{\partial}{\partial x} u_{\lambda_m}(t, X_{\lambda_m}^N(t))$. Using (3.16), we have

$$EI_{\lambda_m}^{k,N}(t) = I_{1,\lambda_m}^{k,N}(t) + I_{2,\lambda_m}^{k,N}(t). \quad (3.21)$$

First we study $I_{1,\lambda_m}^{k,N}(t)$. Using (3.4), we know that $\frac{\partial}{\partial x} u_{\lambda_m} \in C^{1,2}$. So using Itô's formula and Lemma 3.4, we have

$$\begin{aligned} d \frac{\partial}{\partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) &= \left\{ \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) ds \\ &\quad + \gamma_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) dW_{\lambda_m}(s) \end{aligned} \quad (3.22)$$

Using this equation, Lemma 3.4 and the Itô formula we deduce

$$\begin{aligned} d \left[\beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \right] &= \left\{ \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right. \\ &\quad \left. + z_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) ds \\ &\quad + \left\{ \beta_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} + z_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) dW_{\lambda_m}(s). \end{aligned}$$

Integrating between t and t_{k+1} , taking expectation, and using the fact that $\beta_{\lambda_m}^{k,N}(t_{k+1}) = -\lambda_m X_{\lambda_m}^N(t_{k+1})$, so that $I_{1,\lambda_m}^{k,N}(t_{k+1}) = 0$, we obtain

$$\begin{aligned} I_{1,\lambda_m}^{k,N}(t) &= -E \int_t^{t_{k+1}} \left\{ \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right. \\ &\quad \left. + z_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) ds. \end{aligned} \quad (3.23)$$

Using (3.7) and Lemma 3.5, we have for $s \in [t, t_{k+1}]$

$$E\beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) = 4\lambda_m e^{-2\lambda_m(T-s)} E\beta_{\lambda_m}^{k,N}(s) X_{\lambda_m}^N(s) \leq C\lambda_m e^{-2\lambda_m(T-t_{k+1})},$$

and hence

$$\lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E\beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C\lambda_m^{1-p} h^2 e^{-\lambda_m(T-t_{k+1})}.$$

Using Lemma 3.8, and the above inequality, we deduce

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E\beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq Ch^{p+\frac{1}{2}}. \quad (3.24)$$

Using (3.5) and Lemma 3.5, we have for $s \in [t_k, t_{k+1}]$

$$E \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) = 4\lambda_m e^{-2\lambda_m(T-s)} \leq 4\lambda_m e^{-2\lambda_m(T-t_{k+1})},$$

so that

$$\lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C \lambda_m^{1-p} h^2 e^{-2\lambda(T-t_{k+1})}.$$

Thus, Lemma 3.8 yields

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \left| \beta_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C h^{p+\frac{1}{2}}. \quad (3.25)$$

Using equations (3.5) and Lemma 3.4 we have for all $s \in [t, t_{k+1}]$

$$\begin{aligned} E \left| z_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \right| &= \frac{2\lambda_m}{1 + \lambda_m h} \left(1 - \frac{(s - t_k)\lambda_m}{1 + \lambda_m h} \right) e^{-2\lambda_m(T-s)} \\ &\leq C \lambda_m e^{-2\lambda_m(T-t_{k+1})}. \end{aligned}$$

Therefore, we obtain

$$\lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \left| z_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \right| \leq C \lambda_m^{1-p} h^2 e^{-2\lambda_m(T-t_{k+1})}.$$

Using once more Lemma 3.8, we deduce

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \left| z_{\lambda_m}^{k,N}(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \right| \leq C h^{p+\frac{1}{2}}.$$

Plugging this inequality together with (3.24) and (3.25) into (3.23) gives us

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| I_{1,\lambda_m}^{k,N}(t) \right| dt \leq C h^{p+\frac{1}{2}}. \quad (3.26)$$

Now we study $I_{2,\lambda_m}^{k,N}(t)$. Using Lemma 3.4, equation (3.22) and the Itô formula we have

$$\begin{aligned} dX_{\lambda_m}^N(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) &= \left\{ X_{\lambda_m}^N(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + X_{\lambda_m}^N(s) \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right. \\ &\quad \left. + \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} + \left| \gamma_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) ds \\ &\quad + \left\{ \gamma_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} + X_{\lambda_m}^N(s) \gamma_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) dW_{\lambda_m}(s) \end{aligned}$$

So integrating between t and t_{k+1} and taking expectation, we obtain

$$\begin{aligned} I_{2,\lambda_m}^{k,N}(t) &= -\lambda_m E \int_t^{t_{k+1}} \left\{ X_{\lambda_m}^N(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m} + X_{\lambda_m}^N(s) \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right. \\ &\quad \left. + \left| \gamma_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}^N(s)) ds. \end{aligned} \quad (3.27)$$

Using equation (3.7) and Lemma 3.5, we have for all $s \in [t, t_{k+1}]$

$$\begin{aligned} \lambda_m E X_{\lambda_m}^N(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) &= 4\lambda_m^2 e^{-2\lambda_m(T-s)} E \left| X_{\lambda_m}^N(s) \right|^2 \\ &\leq C \lambda_m^2 \left(\frac{1}{\lambda_m} + h \right) e^{-2\lambda_m(T-t_{k+1})}. \end{aligned}$$

Therefore,

$$\lambda_m^{-p} \int_{t_k}^{t_{k+1}} \int_t^{t_{k+1}} \lambda_m E X_{\lambda_m}^N(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C (\lambda_m^{1-p} h^2 + \lambda_m^{2-p} h^3) e^{-2\lambda_m(T-t_{k+1})},$$

and using Lemma 3.8, we deduce

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds \lambda_m E X_{\lambda_m}^N(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C h^{p+\frac{1}{2}}. \quad (3.28)$$

The equation (3.4) and Lemma 3.5 yield for all $s \in [t, t_{k+1}]$

$$\lambda_m E \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) = 2\lambda_m e^{-2\lambda_m(T-s)} E \beta_{\lambda_m}^{k,N}(s) X_{\lambda_m}^N(s) \leq C \lambda_m e^{-2\lambda_m(T-t_{k+1})}.$$

This upper estimate implies

$$\lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds \lambda_m E \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C \lambda_m^{1-p} h^2 e^{-2\lambda_m(T-t_{k+1})},$$

and Lemma 3.8 yields

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds \lambda_m E \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C h^{p+\frac{1}{2}}. \quad (3.29)$$

Using equation (3.5) and Lemma 3.5, we have for all $s \in [t, t_{k+1}]$

$$\lambda_m E X_{\lambda_m}^N(s) \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C \lambda_m e^{-2\lambda_m(T-t_{k+1})}.$$

Therefore, we obtain

$$\lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds \lambda_m E X_{\lambda_m}^N(s) \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C \lambda_m^{1-p} h^2 e^{-2\lambda_m(T-t_{k+1})},$$

and Lemma 3.8 implies

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds \lambda_m E X_{\lambda_m}^N(s) \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C h^{p+\frac{1}{2}}. \quad (3.30)$$

Finally, (3.5) and Lemma 3.4 imply that for all $s \in [t, t_{k+1}]$

$$\lambda_m E \left| \gamma_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C \lambda_m e^{-2\lambda_m(T-t_{k+1})}.$$

This yields

$$\lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds \lambda_m E \left| \gamma_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C \lambda_m^{1-p} h^2 e^{-2\lambda_m(T-t_{k+1})},$$

and Lemma 3.8 implies

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds \lambda_m E \left| \gamma_{\lambda_m}^{k,N}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_{\lambda_m}^N(s)) \leq C h^{p+\frac{1}{2}}.$$

Plugging this inequality together with (3.28) - (3.30) into (3.27), we deduce

$$\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| I_{2,\lambda_m}^k(t) \right| dt \leq C h^{p+\frac{1}{2}}.$$

This equation together with (3.21) and (3.26) conclude the proof. \square

Theorem 2.1 is a straightforward consequence of equation (3.18) and Lemmas 3.9-3.11.

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REFERENCES

- [1] Aboura O., Weak error expansion of the implicit Euler scheme
- [2] Andersson A., Larsson S., Weak convergence for a spatial approximation of the nonlinear stochastic heat equation, arxiv 1212.5564
- [3] Bally V., Talay D., The law of the Euler scheme for stochastic differential equations: I. Convergence rate of the distribution function. *Prob. Th. Rel. Fields*, 104-1:43–60, 1996.
- [4] Da Prato G., Zabczyk J., *Stochastic equations in infinite dimensions*
- [5] de Bouard A., Debussche A., Weak and strong order of convergence of a semi discrete scheme for the stochastic Nonlinear Schrodinger equation, *Applied Mathematics and Optimization journal*, 54, 3, pp. 369–399 2006.
- [6] Debussche A., Weak approximation of stochastic partial differential equations: the nonlinear case, *Math of Comp*, 80, pp. 89–117 2011.
- [7] Debussche A., Printems J., Weak order for the discretization of the stochastic heat equation, *Math of Comp*, 78, 266, pp. 845–863 2009.
- [8] Gyöngy I., Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. I *Potential Anal.* 9, no. 1, 1–25, 1998.
- [9] Gyöngy I., Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II *Potential Anal.* 11, no. 1, 1–37, 1999.
- [10] Hausenblas E., Weak approximation for semilinear stochastic evolution equations, *Stochastic Analysis and Related Topics VIII*, 2003
- [11] Kloeden P. E., Platen E., *Numerical solution of stochastic differential equations*. Applications of Mathematics (New York), 23. Springer-Verlag, Berlin, 1992.
- [12] Kovács M., Larsson S., Lindgren F., Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise, *BIT* **52**, 2012.
- [13] Kovács M., Larsson S., Lindgren F., Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise II. Fully discrete schemes, *BIT Numer. Math.*, 2012.
- [14] Nualart D., *The Malliavin Calculus and Related Topics. Second Edition*. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006.
- [15] Printems J., On the discretization in time of parabolic stochastic partial differential equations, *Math. Model. and Numer. Anal.*, 35(6), 1055-1078, 2001.
- [16] Talay D., Tubaro L., Expansion of the global error for numerical schemes solving SDEs *Stoch. Anal. and App.*, 8(4),483–509, 1990.

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